ON ACCUMULATION OF DISTURBANCES IN NONSTATIONARY LINEAR IMPULSIVE SYSTEMS

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The problem of determining the maximum value of $y_{\max}(T)$ of the particular solution of the differential equation $L\{y(t)\} = f(t)$ subject to the conditions $|f(t)| < M_0$, $0 < t < T$ at instant T has been solved by Bulgakov and Kuzovkova [1, 2]. An analogous problem for linear difference equations has been treated by Roitenberg [3]. However, in a number of cases much more rigid conditions are imposed upon the right-hand side of the equation, i.e. in addition to the constraints on the modulus of $f(t)$ there may be also constraints on the moduli of some of its derivatives such as $f'(t)$, $f''(t)$. This situation may be observed in systems where position, velocity, and acceleration of the controlled object are constrained. In the presence of such additional constraints on the right-hand side of the equation, the magnitude of $y_{\max}(T)$ may be considerably smaller than in the case of a single constraint on $|f(t)|$. In this paper a method of determining the maximum value of $y_{\max}(T)$ of the particular solution of the linear difference equation $L\{y(t)\} = f(t)$ is given for the following cases

$$|f^{(m)}(t)| \leqslant M_m, \qquad m > 0 \tag{0.1}$$

$$|f(t)| \leqslant M_0, \qquad |f'(t)| \leqslant M_1, \qquad |f''(t)| \leqslant M_2 \tag{0.2}$$

1. Let us consider the following equation

$$y(t+n) + P_1(t)\, y(t+n-1) + \ldots + P_n(t)\, y(t) = f(t) \tag{1.1}$$

Its particular solution is of the form

$$y(t) = \sum_{i=0}^{[t]-1} \psi_i(t)\, f(t - [t] + i) \tag{1.2}$$

Hereafter the symbol $[t]$ will denote an integer of time, functions $\psi_i(t)$ will be defined by a linearly independent system of solutions of a homogeneous equation corresponding to equation (1.1).

Let us determine $y_{\max}(T)$ for the following case

$$|f^{(m)}(u)| \leqslant M_m \qquad (m > 0,\ 0 \leqslant u \leqslant T) \tag{1.3}$$

For simplicity let us assume that

$$f(0) = f'(0) = \cdots\cdots = f^{(m-1)}(0) = 0 \tag{1.4}$$

The general case may be treated in an analogous manner. Taking into account (1.4), we have

$$f(u) = \int_0^u \frac{1}{(m-1)!} f^{(m)}(z)(u-z)^{m-1} dz \tag{1.5}$$

Substituting (1.5) into (1.2), we obtain

$$y(T) = \sum_{i=0}^{[T]-1} \psi_i(t) \int_0^{T-[T]+i} \frac{1}{(m-1)!} f^{(m)}(u)(T-[T]+i-u)^{m-1} du =$$
$$= \int_0^{T-1} f^{(m)}(u) \sum_{i=0}^{[T]-1} \frac{1}{(m-1)!} A_i(u)(T-[T]+i-u)^{m-1} du = \int_0^{T-1} f^{(m)}(u) F_m(u) du \tag{1.6}$$

Here

$$A_i(u) = \begin{cases} \psi_i(T), & \text{if } u \in [0, T-[T]+i] \\ 0 & \text{if } u \in [T-[T]+i,\ T-1] \end{cases} \tag{1.7}$$

$$F_m(u) = \sum_{i=0}^{[T]-1} \frac{1}{(m-1)!} A_i(u)(T-[T]+i-u)^{m-1} \tag{1.8}$$

Function $F_m(u)$ has a finite number of discontinuities in the interval $[0,\ T-1]$ and is bounded. It follows from (1.6) that $y(T)$ will assume maximum value permitted by (1.3) at the time T, if the following is true

$$|f_{(m)}(u)| = M_m, \qquad \operatorname{sign} f^{(m)}(u) = \operatorname{sign} F^m(u), \qquad u \in [0,\ T-1] \tag{1.9}$$

2. Let us seek maximum possible value $y_{\max}(T)$ of the particular solution of equation (1.1) at the fixed time T, if the right-hand side of (1.1) is subject to the following constraints:

$$|f(u)| \leqslant M_0, \qquad |f'(u)| \leqslant M_1, \qquad |f''(u)| \leqslant M_2 \tag{2.1}$$

where M_0, M_1, and M_2 are arbitrary constants. Let us note in passing that the problems of determining $y_{\max}(T)$ for the conditions

$$|f(u)| \leqslant M_0, \quad |f'(u)| \leqslant M_1 \tag{2.2}$$

or for the conditions

$$|f(u)| < M_0, \qquad |f''(u)| \leqslant M_2 \tag{2.3}$$

are special cases of the problem under consideration.

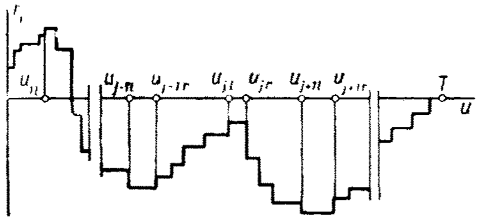

Fig. 1.

The problem of determining $y_{max}(T)$ for the condition

$$|f'(u)| \leqslant M_1, \qquad |f''(u)| \leqslant M_2 \tag{2.4}$$

is reduced to the problem of determining $y_{max}(T)$ for the condition (2.2).

As before, for the sake of simplicity, the assumption is made that $f(0) = f'(0) = 0$. Taking $m = 1$ in (1.6), we obtain

$$y(T) = \int_0^{T-1} F_1(u) f'(u)\, du, \quad F_1(u) = \sum_{i=0}^{[T]-1} A_i(u) \tag{2.5}$$

Function $F_1(u)$ is constant over each half-interval $(T - [T] + i,\ T - [T] + i + 1)$ $(i = 0, 1, \ldots, [T] - 2)$ and over the interval $[0, T-[T]]$. Let the interval $[0, T - 1]$ possess k intervals $(u_{j,l}, u_{j,r})$ $(j = 1, \ldots, k)$ where $F_1(u)$ assumes maximum and minimum values as compared with its values over the intervals on either side (Fig. 1). It is assumed that the following conditions are satisfied:

$$u_{j+1,\,l} - u_{jl} \geqslant 2L_1 \qquad (j = 1, \ldots, k), \qquad u_{1l} \geqslant L_1 + L_2 \tag{2.6}$$

where

$$u_{k+1,\,l} = T - 1$$

$$L_1 = \frac{2M_0}{M_1} + \frac{2M_1}{M_2} \quad \text{for } \frac{M_1^2}{2M_2} < M_0, \qquad L_1 = 2\sqrt{\frac{2M_0}{M_2}} \quad \text{for } \frac{M_1^2}{2M_2} \geqslant M_0$$

$$L_2 = \frac{M_0}{M_1} + \frac{M_1}{M_2} \quad \text{for } \frac{M_1^2}{M_2} < M_0, \qquad L_2 = 2\sqrt{\frac{M_0}{M_2}} \quad \text{for } \frac{M_1^2}{M_2} \geqslant M_0$$

Consequently, for the given equation (1.1) we consider only those values of M_1 and M_2 which satisfy the above inequalities for a fixed value of M_0. However, even in this case (constraint on $|f'(u)|$ and $|f''(u)|$) the magnitude of the maximum accumulated error may be considerably smaller than in the case of a single constraint on $|f(u)|$.

Conditions (2.6) permit one to solve rather simply the following degenerate variational problem: among functions of class A (these functions satisfy conditions (2.1) and their second derivatives over the interval $[0, T-1]$ may have a finite number of discontinuities of the first kind) find function $f_{\max}(u)$ such that the functional (2.5) is maximum. Construction of $f_{\max}(u)$ (thereafter to be called maximum function) is carried out in the following fashion: an arbitrary function of class A is transformed step-by-step in such a manner that the transformed function would remain in the class A and the functional (2.5) would increase. The function obtained in the end of this process shall be independent of the choice of $f(u)$ and shall be maximum. Let us first introduce the following definition.

Let there be given functions $\phi(u)$ of class E in the interval $[u_0, \infty]$ and satisfying the following conditions:

$$(1)\ |\varphi'(u)| \leqslant M_1, \quad |\varphi''(u)| \leqslant M_2; \quad (2)\ \varphi(u_0) = b_0, \quad \varphi'(u_0) = b_{0,1}$$
$$(3)\ \varphi(u_\varphi) = b_1, \quad \varphi'(u_\varphi) = 0 \quad (4)\ \varphi(u) \equiv b_1 \quad \text{for } u > u_\varphi$$

Here the point u_0 is fixed and the point u_ϕ depends on the choice of functions $\phi(u)$.

Let to every pair of values $\phi(u)$, $\phi'(u)$ correspond a point $B(\phi(u), \phi'(u))$ in the N plane. Function $\phi_1(u)$ of class E will be called the optimum function which in the interval $[u_0, u_{\phi 1}]$ realizes the transfer from the point $B_0(b_0, b_{01})$ to the point $B_1(b_1, 0)$, if for the point $u_{\phi 1}$, corresponding to $\phi_1(u)$, the following inequality is satisfied $u_{\phi 1} - u_0 \leqslant u_\phi - u_0$ where u_ϕ corresponds to an arbitrary function $\phi(u)$ of class E. Existence, uniqueness, and construction method of the optimum function $\phi_1(u)$ are shown in [5,6].

In the interval $[-\infty, u_0]$ there also exists a unique optimum function $\phi_1(u)$ which in the interval $[u_0, u_{\phi 1}]$ (where $u_{\phi 1} < u_0$) realizes the transfer from the point $B(b_0, b_{01})$ to the point $B_1(b_1, 0)$.

In order to distinguish between these two optimum functions we will specify the sign of the expression $u_{\phi 1} - u_0$.

Let us now consider one of the segments $[u_{jl}, u_{j+1,l}]$. To simplify the notation let us discard the index l.

In order to be specific, let $F_1(u_{j+1}) > F_1(u_j)$. Let us determine the function $g_j(u)$ in the interval $[u_j, u_{j+1}]$ (see Fig. 2). In the interval $[u_j, u_{j,1}]$ where $u_{j1} - u_j \geqslant 0$ the function $g_j(u)$ shall be the optimum function realizing the transfer from the point $B_j(f(u_j), f'(u_j))$ to the point $B^-(-M_0, 0)$ of the N plane. In the interval $[u_{j+1,0}, u_{j+1}]$ where

$u_{j+1,0} \leqslant u_{j+1}$, $g_j(u)$ is the optimum function realizing the transfer from the point $B_{j+1}(f(u_{j+1}), f'(u_{j+1}))$ to the point $B^-(-M_0, 0)$. In the interval $[u_{j1}, u_{j+1,0}]$ the identity $g_j(u) \equiv -M_0$ holds. Taking into account that $g_j(u_j) = f(u_j)$, $g_j(u_{j+1}) = f(u_{j+1})$ and that $f(u)$ belongs to the class A, it is not difficult to show that $u_{j,1} - u_j \leqslant L_1$, $u_{j+1} - u_{j+1,0} \leqslant L_1$. Since $u_{j+1} - u_j \geqslant 2L_1$, the conditions introduced during the definition of $g_j(u)$ are not contradictory. It also follows that $|g_j(u)| \leqslant M_0$ in the interval $[u_j, u_{j+1}]$.

Let us now change the function $f(u)$ to $g_j(u)$ in the interval $[u_j, u_{j+1}]$ and prove that

$$K^{(j)} = \int_{u_j}^{u_{j+1}} F_1(u)\, g_j'(u)\, du - \int_{u_j}^{u_{j+1}} F_1(u) f'(u)\, du \geqslant 0 \tag{2.7}$$

Expressions (2.7) may be written as follows:

$$\int_{u_j}^{u_{j1}^*} [g_j'(u) - f'(u)]\, F_1(u)\, du + \int_{u_{j1}^*}^{u_{j1}} [g_j'(u) - f'(u)]\, F_1(u)\, du - \int_{u_{j1}}^{u_{j+1,0}} f'(u)\, F_1(u)\, du + $$
$$+ \int_{u_{j+1,0}}^{u_{j+1,0}^*} [g_j'(u) - f'(u)]\, F_1(u)\, du + \int_{u_{j+1,0}^*}^{u_{j+1}} [g_j'(u) - f'(u)]\, F_1(u)\, du \tag{2.8}$$

Here $u_j \leqslant u_{j1}^* \leqslant u_{j1}$. If $u_j < u_{j1}^* < u_{j1}$, then the curves $g_j'(u)$ and $f'(u)$ intersect at the point $u_{j,1}^*$. From the properties of the optimum functions [5,6] it follows that in the interval $[u_j, u_{j1}]$ there can be no more than one point of intersection of the curves $g_j'(u)$ and $f'(u)$ and, furthermore, that in the interval $[u_j, u_{j1}^*]$ $g_j'(u) - f'(u) \leqslant 0$. An analogous remark can be made regarding the point $u_{j+1,0}^*$; furthermore, in the interval $[u^*_{j+1,0}, u_{j+1}]$ $g_j'(u) - f'(u) > 0$. For the first, second, fourth, and the fifth integrals of the right-hand side of (2.8) the conditions necessary for the application of the mean value theorem are satisfied. This theorem is not applicable directly to the third integral. However, dividing $(u_{j1}, u_{j+1,0})$ into the intervals over which $f'(u)$ does not change sign, and applying the mean value theorem over these intervals, it is possible to obtain the following inequality

$$-\int_{u_{j1}}^{u_{j+1,0}} F_1(u) f'(u)\, du \leqslant -F_1(u_3^*) \int_{u_{j1}}^{u_{j+1,0}} f'(u)\, du \qquad (u_{j1} < u_3^* < u_{j+1,0})$$

Applying the mean value theorem to other integrals we obtain

$$K^{(j)} \geqslant \sum_{i=1}^{5} F_1(u_i^*)\, K_i^{(j)} \tag{2.9}$$

where $K_i^{(j)}$ is the i-th integral from (2.8) if $F_1(u) \equiv 1$

$$u_1^* \in (u_j, u_{j1}^*), \quad u_2^* \in (u_{j1}^*, u_{j1}), \quad u_3^* \in (u_{j1}, u_{j+1,0})$$
$$u_4^* \in (u_{j+1,0}, u_{j+1,0}^*), \quad u_5^* \in (u_{j+1,0}^*, u_{j+1})$$

and, consequently,

$$F_1(u_1^*) \leqslant F_1(u_2^*) \leqslant F_1(u_3^*) \leqslant F_1(u_4^*) \leqslant F_1(u_5^*) \tag{2.10}$$

Since the boundary conditions for the functions $g_i(u)$ and $f(u)$ coincide at the points u_j and u_{j+1}, then it follows that

$$K_1^{(j)} + \ldots + K_5^{(j)} = 0 \tag{2.11}$$

It is obvious that

$$K_1^{(j)} + K_2^{(j)} \leqslant 0, \qquad K_4^{(j)} + K_5^{(j)} \geqslant 0 \tag{2.12}$$

and from (2.9) to (2.12) it follows that $K^{(j)} \geqslant 0$.

If the inequality $F_1(u_j) > F_1(u_{j+1})$ were satisfied, then in the intervals $[u_j, u_{j1}]$ and $[u_{j+1,0}, u_{j+1}]$ the function $g_j(u)$ would be defined as the optimum function realizing the transfer from the point $B_j(f(u_j), f'(u_j))$ to the point $B^+(M_0, 0)$ on the N plane in the interval $[u_j, u_{j1}]$; and as the optimum function realizing the transfer from the point $B_{j+1}(f(u_{j+1}), f'(u_{j+1}))$ to the point $B^+(M_0, 0)$ in the interval $[u_{j+1}, u_{j+1,0}]$; in the interval $(u_{j1}, u_{j+1,0})$ we shall have $g_j(u) \equiv M_0$; in this case $K^{(j)} \geqslant 0$.

Let us now define the function $g(u)$ in the interval $[0, T - 1]$ as follows. The function $g(u) = g_j(u)$, if u lies in the interval $[u_j, u_{j+1}]$ ($j = 0, 1, \ldots, k$). It is obvious that $g(u)$ belongs to the A class of the functions under consideration, and since $K^{(j)} > 0$ for any interval $[u_j, u_{j+1}]$, then replacement of $f(u)$ by $g(u)$ in the interval $[0, T - 1]$ can only increase the functional (2.5). If $F(u_j) < F(u_{j+1})$ then changing $g(u)$ will result in the corresponding transfer of the point $B(g(u), g'(u))$ on the N plane from ghe point $B^+(M_0, 0)$ to the point $B^-(-M_0, 0)$. Let us note that the point u_{j0} in the interval (u_{j-1}, u_j) is defined (Fig. 2) analogously to the point $u_{j+1,0}$ in the interval (u_j, u_{j+1}).

Let $E(u - c_{j0})$ be the optimum function realizing the transfer from the point $B^+(M_0, 0)$ to the point $B^-(-M_0, 0)$ of the N plane in the interval $[c_{j0}, c_{j1}]$. This function is easy to construct.

Expression for $E''(u - c_{j0})$ over the interval $[c_{j0}, c_{j1}]$ for the case when $1/2\; M_1^2/M_2 < M_0$ is of the form

$$E''(u - c_{j0}) = -M_2 \quad \text{for } u \text{ on } \left[c_{j0},\ c_{j0} + \frac{M_1}{M_2}\right]$$

$$E''(u - c_{j0}) = 0 \quad \text{for } u \text{ on } \left[c_{j0} + \frac{M_1}{M_2},\ c_{j0} + \frac{2M_0}{M_1}\right]$$

$$E''(u - c_{j0}) = M_2 \quad \text{for } u \text{ on } \left[c_{j0} + \frac{2M_0}{M_1},\ c_{j0} + \frac{2M_0}{M_1} + \frac{M_1}{M_2}\right]$$

The formula is just as simple for the case when $1/2\ M_1^2/M_2 > M_0$. $E(u - c_{j0})$ is the optimum function and, consequently, $c_{j1} - c_{j0} \leqslant u_{j1} - u_{j0}$. Since $F_1(u)$ does not increase in the interval $(u_{j0},\ u_j)$ and does not decrease in the interval $(u_j,\ u_{j1})$, then it is possible to choose the interval $(c_{j0},\ c_{j1})$ in the interval $(u_{j0},\ u_{j1})$ such that any value of the function $F_1(u)$ over this interval would not be less than the values of $F_1(u)$ over the intervals $(u_{j0},\ c_{j0})$ and $(c_{j1},\ u_{j1})$.

If $F_1(u_j) > F_1(u_{j+1})$, then changing $g(u)$ in the interval $[u_{j0},\ u_{j1}]$ will result in the corresponding transfer of the point $B(g(u),\ g'(u))$ from the point $B^-(-M_0,\ 0)$ to the point $B^+(M_0,\ 0)$. Let $E_1(u - c_{j0})$ denote the optimum function realizing the transfer from the point $B^-(-M_0,\ 0)$ on the N-plane in the interval $[c_{j0},\ c_{j1}]$.

It is obvious that $E_1(u - c_{j0}) = -E(u - c_{j0})$, and that it is possible to choose the subinterval $(c_{j0},\ c_{j1})$ in the interval $(u_{j0},\ u_{j1})$ such that any value of $F_1(u)$ over this subinterval would not be greater than any value of $F_1(u)$ over the intervals $(u_{j0},\ c_{j0})$ and $(c_{j1},\ u_{j1})$. Let us replace the function $g(u)$ in the interval $[0,\ T-1]$ by the function $h(u)$ which is defined as follows:

$$h(u) = \pm E(u - c_{j0}) \text{ on } (c_{j0}, c_{j1}),\ h(u) = \pm M_0 \text{ on } (u_{j0}, c_{j0}),\ h(u) = \mp M \text{ on } (c_{j1}, u_{j1}),$$

here the upper signs refer to $F_1(u_j) < F_1(u_{j+1})$, while the lower signs refer to $F_1(u_j) > F_1(u_{j+1})$.

If the point u does not belong to either of the intervals $(u_{j0},\ u_{j1})$, then we set $h(u) = g(u)$ (at these points $|h(u)| = M_0$). Since $h(u)$ belongs to the A class of the functions under consideration, then by applying the mean value theorem, it is easy to show that the replacement of $f(u)$ by $h(u)$ can only increase the functional (2.5). Now in order to increase $y(T)$ we can vary the position of the points c_{j0} $(j = 1, \ldots, k)$.

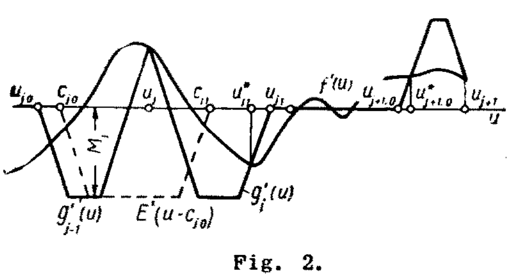

Fig. 2.

In order to determine the point c_{j0} we must maximize the following expression

$$\Phi(c_{j0}) = \int_{c_{j0}}^{c_{j0}+L} F_1(u)\, E'(u - c_{j0})\, du \qquad \left(L = \frac{2M_0}{M_1} + \frac{M_1}{M_2}\right) \tag{2.13}$$

The point c_{j0} is the root of the equation

$$\Phi'(c_{j0}) = -\int_{c_{j0}}^{c_{j0}+L} F_1(u)\, E''(u - c_{j0})\, du = 0 \tag{2.14}$$

Substituting the expression for $E''(u - c_{j0})$ into (2.14), we obtain (2.15)

$$-Z\left(c_{j0} + \frac{M_1}{M_2}\right) + Z(c_{j0}) + Z(c_{j0} + L) - Z\left(c_{j0} + \frac{2M_0}{M_1}\right) = 0; \qquad Z(u) = \int_0^u F_1(u)\, du$$

It is easy to show that in the interval $(u_j - L,\ u_j + L)$ this equation will have at least one real root and that the expression (2.13) will attain the same value for any real root lying in this interval. Let us denote the function $h(u)$ with c_{j0} as the roots of (2.14) through $f_{\max}(u)$; this is the maximum function. Let us note that the maximum function may not be unique but that the functional (2.5) will attain the same value for any maximum function.

Taking into consideration the form of $f_{\max}(u)$ and applying the mean value theorem, $y_{\max}(T)$ may be represented as follows:

$$|y_{\max}(T)| = |F_1(u_0^*) M_0 - \sum_{j=1}^{k} F_1(u_j^*)\, 2M_0 (-1)^{j-1}| \tag{2.16}$$

Here

$$u_0^* \in (0,\ L_2)|, \qquad u_j^* \in (c_{j0},\ c_{j1})|, \qquad c_{j1} - c_{j0} = L.$$

If $M_2 \to \infty$, $M_1 \to \infty$, then $L_2 \to 0$, $L \to 0$, and consequently, in this case we have

$$|y_{\max}(T)| = |F_1(0) M_0 - \sum_{j=1}^{k} F_1(u_{j,l})\, 2M_0 (-1)^{j+1}| = M_0 \sum_{j=1}^{[T]-1} |\psi_j| \tag{2.17}$$

(the last equality is obtained on the basis of (1.7) and (1.8) and in this fashion we arrive at the case investigated by Roitenberg [3]).

Remark 1. The above method of constructing maximum function also applies to linear differential equations.

Remark 2. It is obvious that the above method permits one to determine $y_{\max}(T)$ in the following nonlinear automatic control system

$$x_1' = F_0(f(t) - x_4), \qquad x_2^{\bullet} = F_1(x_1), \qquad x_3^{\bullet} = F_2(x_2)$$
$$L_1\{y(t)\} = x_3, \qquad L_2\{x_4\} = y$$

Here

$$F_i(u) = u \text{ for } |u| \leqslant M_i, \qquad F_i(u) = M_i \quad \text{for } |u| > M_i$$

L_1 and L_2 are linear differential operators, the dots denote derivatives with respect to time.

Example. Let us determine the maximum value $y_{max}(T)$ for $T = 13$ of the particular solution of the following equation

$$y(t+2) + 1.75\, y(t+1) + y(t) = f(t) \tag{2.18}$$

for the following cases

$$(1)\; |f(u)| \leqslant M_0, \quad |f'(u)| \leqslant M_1, \quad |f''(u)| \leqslant M_2, \quad (2)\; |f(u)| \leqslant M_0. \quad \begin{pmatrix} M_0 = 3 \\ M_1 = 2 \\ M_2 = 13 \end{pmatrix}$$

Using the above method, we find that in the first case $y_{max}(T) = 19$, in the second case $y_{max}(T) = 24$. Thus, if the constraints on $f'(u)$ and $f''(u)$ are not taken into account the value of $y_{max}(T)$ increases by 25 per cent.

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